

Concise Quantum Associative Memories with Nonlinear Search Algorithm

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Abstract

The model of quantum associative memories proposed here is quietly similar to that of Rigui Zhou *et al.* [1] based on quantum matrix with binary decision diagram and nonlinear search algorithm put forth by David Rosenbaum [2] and Abrams and Lloyd [3] respectively. Our model, that simplify and generalize that of Ref. [1], gives the possibility to retrieve one of desired states in multi-values retrieving, when a measure on the first register is done. If n is the number of qubit of first register, $p \leq 2^n$ the number of stored patterns, $q \leq p$ the number of stored patterns if the value of t are known (i.e., t qubits have been measure or are already be disentangled to others, or the oracle acts on a subspace of $(n-t)$ qubits), $m \leq q$ the number of values x for which $f(x) = 1$, $c = \lceil \log_2 q \rceil$ the least integer greater or equal to $\log_2 q$, and $r = \lfloor \log_2 m \rfloor$ the integer part of $\log_2 m$, then the time complexity of our algorithm is $\mathcal{O}(c-r)$. It is better than Grover's algorithm and the modified forms which need $\mathcal{O}(\sqrt{\frac{2^n}{m}})$ steps, when they are used as the retrieval algorithm.

1 Introduction

Quantum Neural Networks are Artificial Neural Networks functioning according to quantum laws. One of the useful Neural Networks is the Associative Memory, which is an important tool for pattern recognition, intelligent control and artificial intelligence. Ventura and Martinez have built a model of Quantum Associative Memory where the stored patterns are considered to be basis states of the memory quantum

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state [4]. They used a modified version of the well known Grover's quantum search algorithm in an unsorted database as the retrieval algorithm. In order to overcome the limitation of that model to only solve the completion problem by doing data retrieving from noisy data, Ezhov *et al.* have used an *exclusive* method of quantum superposition and Grover's algorithm with distributed queries [5]. However, their model still produces non-negligible probability of irrelevant classification. Recently, we have put forth an improved model of Quantum Associative Memory with distributed query that reduce the probability of this irrelevant classification [6].

The nonlinear search algorithm is based on the fact that it has been suggested that, under some circumstances, the superposition principle of quantum mechanics might be violated. In other words, sometime a quantum system might have temporal nonlinear evolution. Therefore nonlinear quantum computer could solve NP-complete and even #P problems in polynomial time argued in 1998 Abrams and Llyod in their nowadays classic paper [3]. A simplified version was suggested the same year by Czachor in [7]. Rigui *et al.* [1] have recently proposed a model of Ventura's associative memory, which use Binary Superposed Quantum Decision Diagram (BSQDD) as learning process and the above nonlinear algorithm of Abrams and Llyod as retrieving process, for multi-values retrieval. Although the learning process of their model is good, there is nevertheless some ambiguities on how evolves the memory and how the multi-values retrieval arises. Here we propose a concise Nonlinear Search Algorithm for Quantum Associative Memories, with a method to retrieve one of desired states, especially in multi-values retrieving, when a measure on the first register is done, and not on the flag qubit. Thus our model simplify and generalize that of Rigui *et al.* [1]. If n is the number of qubit of first register, $p \leq 2^n$ the number of stored patterns, $q \leq p$ the number of stored patterns if the value of t are known (i.e., t qubits have been measure or are already be disentangled to others, or the oracle acts on a subspace of $(n - t)$ qubits), $m \leq q$ the number of values x for which $f(x) = 1$, $c = \text{ceil}(\log_2 q)$, i.e., the least integer greater or equal to $\log_2 q$, and $r = \text{int}(\log_2 m)$ the integer part of $\log_2 m$, then the time complexity of our algorithm is $\mathcal{O}(c - r)$.

The paper is organized as follows: section 2 describes without any ambiguities or complications the nonlinear search algorithm proposed by Abrams and Llyod. Section 3 presents Quantum Associative Memories with Nonlinear Search Algorithm, with a new method to retrieve one of desired states in multi-values retrieving. Concrete examples of simulation are given to illustrate the theory in each section. Finally, a short conclusion is provided in section 4.

2 Nonlinear search algorithm

Suppose there is an unitary transformation U_f , the oracle or the *black box*, which act as follows: for a set of inputs between 0 and $2^n - 1$, there is at most one x for which $f(x) = 1$ and the other values give 0. Let us consider two registers; the first register, which is a n -qubit system, is to compute inputs and the second, which is a single-qubit system, is to compute the answer of the oracle. We can define the function f like:

$$\begin{aligned} f : \mathcal{H}^{\otimes 2n} &\longrightarrow \mathcal{H}^2 \\ |y\rangle &\longmapsto |\delta_{xy}\rangle \end{aligned} \tag{1}$$

where $\mathcal{H}^{\otimes 2n}$ is a Hilbert space of $2n$ dimensions.

The main goal of Abrams and Llyod nonlinear algorithm, summarizes by Algorithm 1, is to disentangle flag qubit from the first register, as a measure on the flag qubit can tell us if there is at most a value x

for which $f(x) = 1$; this is done by transforming the part of the flag qubit which is $|0\rangle$ to $|1\rangle$. They claim that it is not possible to do using linear operators of quantum information.

Algorithm 1 Nonlinear search algorithm

- 1: Put the first register in the superposed state of all the N values and the flag qubit to $|0\rangle$
 - 2: Apply the oracle U_f
 - 3: **for** each qubit of the first register with the flag qubit **do**
 - 4: Apply the unitary operator U
 - 5:
 1. apply the nonlinear operator NL^-
 2. apply the nonlinear operator NL^+
 - 6: Apply the Hadamard operator W on the qubit of the first register and the NOT operator X on the flag qubit
 - 7: **end for**
 - 8: Observe the flag qubit
-

Let $|\psi\rangle$ be the state which describes all the system, and assume that all $N = 2^n$ inputs are computed in the first register with equal amplitude:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_y^N |y\rangle |0\rangle. \quad (2)$$

When the oracle is apply we have

$$\begin{aligned} U_f |\psi\rangle &= \frac{1}{\sqrt{N}} \sum_y^N |y\rangle |f(y)\rangle \\ &= \frac{1}{\sqrt{N}} \left(\sum_{y \neq x}^N |y\rangle |0\rangle + |x\rangle |1\rangle \right). \end{aligned} \quad (3)$$

To describe the disentanglement algorithm we consider the binary forms of values and assume that there is at most one value x which gives $f(x) = 1$. Let $|j_n j_{n-1} \dots j_1\rangle$ and $|i_n i_{n-1} \dots i_1\rangle$ be the binary forms of states $|y\rangle$ and $|x\rangle$ respectively, with $j_k, i_k \in \{0, 1\}$. Equations (2) and (3) can be rewritten as

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \left[\sum_{\substack{j_n j_{n-1} \dots j_1 = 0 \\ j_n j_{n-1} \dots j_1 \neq i_n i_{n-1} \dots i_1}}^1 |j_n j_{n-1} \dots j_1\rangle + |i_n i_{n-1} \dots i_1\rangle \right] |0\rangle, \quad (4)$$

and

$$U_f |\psi\rangle = |\Psi\rangle = \frac{1}{\sqrt{2^n}} \left[\sum_{\substack{j_n j_{n-1} \dots j_1 = 0 \\ j_n j_{n-1} \dots j_1 \neq i_n i_{n-1} \dots i_1}}^1 |j_n j_{n-1} \dots j_1\rangle |0\rangle + |i_n i_{n-1} \dots i_1\rangle |1\rangle \right]. \quad (5)$$

Highlighting the least significant qubit (LSQ) of the first register, equation (5) can be helpfully written as

$$|\Psi\rangle = \frac{1}{\sqrt{2^n}} \left[\sum_{\substack{j_n j_{n-1} \dots j_1 = 0 \\ j_n j_{n-1} \dots j_2 \neq i_n i_{n-1} \dots i_2}}^1 |j_n j_{n-1} \dots j_1\rangle |0\rangle + |i_n i_{n-1} \dots (1 - i_1)\rangle |0\rangle + |i_n i_{n-1} \dots i_1\rangle |1\rangle \right]. \quad (6)$$

The state (6) must be view as the general binary form of the system after the action of the oracle. It summarizes all particulars states gives by Czachor in [7] and erases ambiguities give by his notation. Indeed, his equation

$$\frac{1}{\sqrt{2^n}} \sum_{j_n j_{n-1} \dots j_2 = 0}^1 [|j_n j_{n-1} \dots 0_1\rangle |1\rangle + |j_n j_{n-1} \dots 1_1\rangle |0\rangle], \quad (7)$$

suggests that there is $2^{n-1} x$ for which $f(x) = 1$, and not $s = 1$ as he claims.

Considering the subsystem of only the LSQ of the first register $|\ell\rangle$ and the flag qubit $|k\rangle$ ($k, \ell \in \{0, 1\}$), the computer will be in one of the following states, where we ignore the normalization constants,

$$|00\rangle + |10\rangle, \quad (8a)$$

$$|10\rangle + |01\rangle, \quad (8b)$$

$$|00\rangle + |11\rangle. \quad (8c)$$

The left part of equation (6) suggest that the state (8a) occur with the highest probability, whereas the state $|01\rangle + |11\rangle$ does not appear because the variable x is suppose to be unique.

The nonlinear evolution (NLE), (step 4 to step 6 of Algorithm 1), aims to transform the states (8b) and (8c) to $|01\rangle + |11\rangle$ while leaving the state (8a) unchanged. The NLE part of the algorithm then acts as follows:

Step 4. Apply the 2-qubit operator

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \quad (9)$$

on the states (8):

$$U(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle - |10\rangle + |11\rangle), \quad (10a)$$

$$U(|10\rangle + |01\rangle) = \sqrt{2}|01\rangle, \quad (10b)$$

$$U(|00\rangle + |11\rangle) = \sqrt{2}|00\rangle. \quad (10c)$$

Step 5.1. Apply the nonlinear operator NL^- on the previous results:

$$NL^- \left[\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \right] = \sqrt{2}|0\rangle(\alpha|0\rangle + \beta|1\rangle), \quad (11a)$$

$$NL^-(\sqrt{2}|01\rangle) = \sqrt{2}|00\rangle, \quad (11b)$$

$$NL^-(\sqrt{2}|00\rangle) = \sqrt{2}|00\rangle. \quad (11c)$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. As we see on the state (11a), the action of the nonlinear operator NL^- on the state $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ is not specify. This gives some flexibility to choose the nonlinear gate NL^- [3]. On the states (11b) and (11c), the operator NL^- maps both $|0\rangle$ and $|1\rangle$ to the state $|0\rangle$. Thus it must be seen as the NOT gate X in case of the state (11b) and the identity gate in case of the state (11c).

Step 5.2. Apply a second 1-qubit nonlinear operator NL^+ on the flag qubit:

$$NL^+[\sqrt{2}|0\rangle(\alpha|0\rangle + \beta|1\rangle)] = \sqrt{2}|01\rangle, \quad (12a)$$

$$NL^+(\sqrt{2}|00\rangle) = \sqrt{2}|00\rangle. \quad (12b)$$

The nonlinear operator NL^+ is the identity gate on the state $|0\rangle$. The general form of the unitary matrix NL^+ which transforms the generic one-qubit $\alpha|0\rangle + \beta|1\rangle$ to $|1\rangle$ is

$$M = \begin{pmatrix} \mp\gamma\beta & \pm\gamma\alpha \\ \alpha^* & \beta^* \end{pmatrix}, \quad \gamma = \pm i \text{ or } \pm 1 \ (i^2 = -1), \quad (13)$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

It noteworthy that Rigui *et al.* [1] claim that matrix M must be

$$V = \begin{pmatrix} 1 & \frac{1}{\beta} \\ 0 & -\frac{\alpha}{\beta} \end{pmatrix}, \quad (14)$$

which is unfortunately not an unitary matrix like matrix (13) as required by quantum information processing. Furthermore, the matrix V yields to a wrong result

$$V[\sqrt{2}|0\rangle(\alpha|0\rangle + \beta|1\rangle)] = \sqrt{2}|0\rangle[(\alpha + 1)|0\rangle - \alpha|1\rangle], \quad (15)$$

and not $\sqrt{2}|01\rangle$ as expected.

Step 6. Apply NOT gate X on flag qubit and Hadamard gate W on the first qubit.

We summarize below the nonlinear evolution of states of equations (8) with their corresponding circuits:

$$|00\rangle + |10\rangle \xrightarrow{U} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \xrightarrow{NL^-} \sqrt{2}|0\rangle(\alpha|0\rangle + \beta|1\rangle) \xrightarrow{I \otimes NL^+} \sqrt{2}|01\rangle \xrightarrow{W \otimes X} |00\rangle + |10\rangle \quad (16a)$$

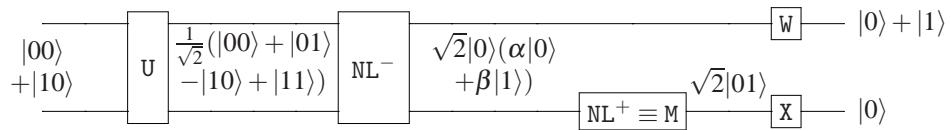


Figure 1: Equivalent circuit of nonlinear evolution (16a).

$$|10\rangle + |01\rangle \xrightarrow{U} \sqrt{2}|01\rangle \xrightarrow{\mathbb{I} \otimes \text{NL}^-} \sqrt{2}|00\rangle \xrightarrow{\mathbb{I} \otimes \text{NL}^+} \sqrt{2}|00\rangle \xrightarrow{W \otimes X} |01\rangle + |11\rangle \quad (16b)$$

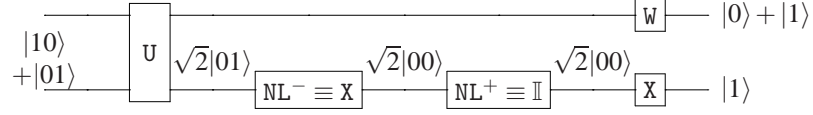


Figure 2: Equivalent circuit of nonlinear evolution (16b).

$$|00\rangle + |11\rangle \xrightarrow{U} \sqrt{2}|00\rangle \xrightarrow{\mathbb{I} \otimes \text{NL}^-} \sqrt{2}|00\rangle \xrightarrow{\mathbb{I} \otimes \text{NL}^+} \sqrt{2}|00\rangle \xrightarrow{W \otimes X} |01\rangle + |11\rangle \quad (16c)$$

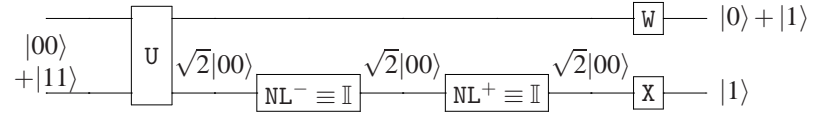


Figure 3: Equivalent circuit of nonlinear evolution (16c).

Example 1. For a better understanding let us consider a simple case of a register with 4-qubits in superposition states of all 16 possibles values, plus a flag qubit. The marked state is $|2\rangle = |0010\rangle$. We start with

$$|\psi\rangle = \frac{1}{4}(|0000\rangle + |0001\rangle + |0010\rangle + |0011\rangle + |0100\rangle + |0101\rangle + |0110\rangle + |0111\rangle + |1000\rangle + |1001\rangle + |1010\rangle + |1011\rangle + |1100\rangle + |1101\rangle + |1110\rangle + |1111\rangle)|0\rangle. \quad (17a)$$

The action of the oracle operator U_f yields to

$$|\Psi\rangle = \frac{1}{4}(|0000\rangle|0\rangle + |0001\rangle|0\rangle + |0010\rangle|1\rangle + |0011\rangle|0\rangle + |0100\rangle|0\rangle + |0101\rangle|0\rangle + |0110\rangle|0\rangle + |0111\rangle|0\rangle + |1000\rangle|0\rangle + |1001\rangle|0\rangle + |1010\rangle|0\rangle + |1011\rangle|0\rangle + |1100\rangle|0\rangle + |1101\rangle|0\rangle + |1110\rangle|0\rangle + |1111\rangle|0\rangle). \quad (17b)$$

Now we will describe as in [7], but without complex details how is the system when the NLE gate is applied.

We start by looking on the LSQ of the register.

$$\begin{aligned}
|\Psi\rangle = \frac{1}{4}(&|000\boxed{0}\rangle|\boxed{0}\rangle + |000\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |001\boxed{0}\rangle|\boxed{1}\rangle + |001\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |010\boxed{0}\rangle|\boxed{0}\rangle + |010\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |011\boxed{0}\rangle|\boxed{0}\rangle + |011\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |100\boxed{0}\rangle|\boxed{0}\rangle + |100\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |101\boxed{0}\rangle|\boxed{0}\rangle + |101\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |110\boxed{0}\rangle|\boxed{0}\rangle + |110\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |111\boxed{0}\rangle|\boxed{0}\rangle + |111\boxed{1}\rangle|\boxed{0}\rangle)
\end{aligned}
\tag{18a}$$

Applying the NLE gate the first time produces:

$$\begin{aligned}
|\Psi_1\rangle = \frac{1}{4}(&|000\boxed{0}\rangle|\boxed{0}\rangle + |000\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |001\boxed{0}\rangle|\boxed{1}\rangle + |001\boxed{1}\rangle|\boxed{1}\rangle \\
&+ |010\boxed{0}\rangle|\boxed{0}\rangle + |010\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |011\boxed{0}\rangle|\boxed{0}\rangle + |011\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |100\boxed{0}\rangle|\boxed{0}\rangle + |100\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |101\boxed{0}\rangle|\boxed{0}\rangle + |101\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |110\boxed{0}\rangle|\boxed{0}\rangle + |110\boxed{1}\rangle|\boxed{0}\rangle \\
&+ |111\boxed{0}\rangle|\boxed{0}\rangle + |111\boxed{1}\rangle|\boxed{0}\rangle)
\end{aligned}
\tag{18b}$$

Next looking on the second LSQ:

$$\begin{aligned}
|\Psi_1\rangle = \frac{1}{4}(&|00\boxed{0}0\rangle|\boxed{0}\rangle + |00\boxed{1}0\rangle|\boxed{1}\rangle \\
&+ |00\boxed{0}1\rangle|\boxed{0}\rangle + |00\boxed{1}1\rangle|\boxed{1}\rangle \\
&+ |01\boxed{0}1\rangle|\boxed{0}\rangle + |01\boxed{1}1\rangle|\boxed{0}\rangle \\
&+ |01\boxed{0}0\rangle|\boxed{0}\rangle + |01\boxed{1}0\rangle|\boxed{0}\rangle \\
&+ |10\boxed{0}1\rangle|\boxed{0}\rangle + |10\boxed{1}1\rangle|\boxed{0}\rangle \\
&+ |10\boxed{0}0\rangle|\boxed{0}\rangle + |10\boxed{1}0\rangle|\boxed{0}\rangle \\
&+ |11\boxed{0}1\rangle|\boxed{0}\rangle + |11\boxed{1}1\rangle|\boxed{0}\rangle \\
&+ |11\boxed{0}0\rangle|\boxed{0}\rangle + |11\boxed{1}0\rangle|\boxed{0}\rangle)
\end{aligned}
\tag{18c}$$

Applying the NLE gate the second time produces:

$$\begin{aligned}
|\Psi_2\rangle = \frac{1}{4}(&|00\boxed{0}0\rangle|\boxed{1}\rangle + |00\boxed{1}0\rangle|\boxed{1}\rangle \\
&+ |00\boxed{0}1\rangle|\boxed{1}\rangle + |00\boxed{1}1\rangle|\boxed{1}\rangle \\
&+ |01\boxed{0}1\rangle|\boxed{0}\rangle + |01\boxed{1}1\rangle|\boxed{0}\rangle \\
&+ |01\boxed{0}0\rangle|\boxed{0}\rangle + |01\boxed{1}0\rangle|\boxed{0}\rangle \\
&+ |10\boxed{0}1\rangle|\boxed{0}\rangle + |10\boxed{1}1\rangle|\boxed{0}\rangle \\
&+ |10\boxed{0}0\rangle|\boxed{0}\rangle + |10\boxed{1}0\rangle|\boxed{0}\rangle \\
&+ |11\boxed{0}1\rangle|\boxed{0}\rangle + |11\boxed{1}1\rangle|\boxed{0}\rangle \\
&+ |11\boxed{0}0\rangle|\boxed{0}\rangle + |11\boxed{1}0\rangle|\boxed{0}\rangle)
\end{aligned}
\tag{18d}$$

Now looking on the third qubit:

$$\begin{aligned}
|\Psi_2\rangle = & \frac{1}{4}(|0\boxed{0}00\rangle|1\rangle + |0\boxed{1}00\rangle|0\rangle \\
& + |0\boxed{0}01\rangle|1\rangle + |0\boxed{1}01\rangle|0\rangle \\
& + |0\boxed{0}10\rangle|1\rangle + |0\boxed{1}10\rangle|0\rangle \\
& + |0\boxed{0}11\rangle|1\rangle + |0\boxed{1}11\rangle|0\rangle \\
& + |1\boxed{0}01\rangle|0\rangle + |1\boxed{1}01\rangle|0\rangle \\
& + |1\boxed{0}11\rangle|0\rangle + |1\boxed{1}11\rangle|0\rangle \\
& + |1\boxed{0}00\rangle|0\rangle + |1\boxed{1}00\rangle|0\rangle \\
& + |1\boxed{0}10\rangle|0\rangle + |1\boxed{1}10\rangle|0\rangle)
\end{aligned}
\tag{18e}$$

Applying the NLE gate the third time produces:

$$\begin{aligned}
|\Psi_3\rangle = & \frac{1}{4}(|0\boxed{0}00\rangle|1\rangle + |0\boxed{1}00\rangle|1\rangle \\
& + |0\boxed{0}10\rangle|1\rangle + |0\boxed{1}10\rangle|1\rangle \\
& + |0\boxed{0}01\rangle|1\rangle + |0\boxed{1}01\rangle|1\rangle \\
& + |0\boxed{0}11\rangle|1\rangle + |0\boxed{1}11\rangle|1\rangle \\
& + |1\boxed{0}01\rangle|0\rangle + |1\boxed{1}01\rangle|0\rangle \\
& + |1\boxed{0}11\rangle|0\rangle + |1\boxed{1}11\rangle|0\rangle \\
& + |1\boxed{0}00\rangle|0\rangle + |1\boxed{1}00\rangle|0\rangle \\
& + |1\boxed{0}10\rangle|0\rangle + |1\boxed{1}10\rangle|0\rangle)
\end{aligned}
\tag{18f}$$

Finally we look on the most significant qubit:

$$\begin{aligned}
|\Psi_3\rangle = & \frac{1}{4}(|\boxed{0}000\rangle|1\rangle + |\boxed{1}000\rangle|0\rangle \\
& + |\boxed{0}001\rangle|1\rangle + |\boxed{1}001\rangle|0\rangle \\
& + |\boxed{0}010\rangle|1\rangle + |\boxed{1}010\rangle|0\rangle \\
& + |\boxed{0}011\rangle|1\rangle + |\boxed{1}011\rangle|0\rangle \\
& + |\boxed{0}100\rangle|1\rangle + |\boxed{1}100\rangle|0\rangle \\
& + |\boxed{0}101\rangle|1\rangle + |\boxed{1}101\rangle|0\rangle \\
& + |\boxed{0}110\rangle|1\rangle + |\boxed{1}110\rangle|0\rangle \\
& + |\boxed{0}111\rangle|1\rangle + |\boxed{1}111\rangle|0\rangle)
\end{aligned}
\tag{18g}$$

Applying the NLE gate the last time produces:

$$\begin{aligned}
|\Psi_4\rangle = & \frac{1}{4}(|\boxed{0}000\rangle + |\boxed{1}000\rangle \\
& + |\boxed{0}001\rangle + |\boxed{1}001\rangle \\
& + |\boxed{0}010\rangle + |\boxed{1}010\rangle \\
& + |\boxed{0}011\rangle + |\boxed{1}011\rangle \\
& + |\boxed{0}100\rangle + |\boxed{1}100\rangle \\
& + |\boxed{0}101\rangle + |\boxed{1}101\rangle \\
& + |\boxed{0}110\rangle + |\boxed{1}110\rangle \\
& + |\boxed{0}111\rangle + |\boxed{1}111\rangle)|1\rangle
\end{aligned}
\tag{18h}$$

A measure on the flag qubit tell us that there is a value (here is 2) which gives $f(x) = 1$.

It appears that we need to apply n times the NLE gate. So, if we know the values of t qubits of our register (i.e., t qubits have been measure or are already disentangled to others, or the oracle acts on a subspace of $(n - t)$ qubits), the NLE gate will be repeat $(n - t)$ times. Let see it with another example, using same conditions.

Example 2. The value of most significant qubit is known and it is $|0\rangle$ (a measure was done on it or

Hadamard gate was applied on it). Our system collapses to:

$$\begin{aligned} |\psi\rangle &= \frac{1}{2\sqrt{2}}(|0000\rangle + |0001\rangle + |0010\rangle + |0011\rangle + |0100\rangle + |0101\rangle + |0110\rangle + |0111\rangle)|0\rangle \\ &= |0\rangle \left[\frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \right] |0\rangle. \end{aligned} \quad (19a)$$

Else the the oracle acts on the 3-qubits and gives 1 for $|010\rangle$, which is a part of the values 2 and 10 in their binary forms (0010 and 1010). The system must be view as:

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \left[\frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \right] |0\rangle. \quad (19b)$$

If the most significant qubit, the fourth qubit, is noted a ($|a\rangle = |0\rangle$ in Eq. (19a) and $|a\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ in Eq. (19b)), the application of oracle operator gives

$$|\Psi\rangle = \frac{1}{2\sqrt{2}}|a\rangle(|000\rangle|0\rangle + |001\rangle|0\rangle + |010\rangle|0\rangle + |011\rangle|1\rangle + |100\rangle|0\rangle + |101\rangle|0\rangle + |110\rangle|0\rangle + |111\rangle|0\rangle) \quad (19c)$$

Next, proceeding like in example 1 without as much as details, each application of the NLE gate on the system gives:

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{2\sqrt{2}}|a\rangle(|010\rangle|1\rangle + |011\rangle|1\rangle \\ &\quad + |000\rangle|0\rangle + |001\rangle|0\rangle \\ &\quad + |100\rangle|0\rangle + |101\rangle|0\rangle \\ &\quad + |110\rangle|0\rangle + |111\rangle|0\rangle) \end{aligned} \quad (19d)$$

$$\begin{aligned} |\Psi_2\rangle &= \frac{1}{2\sqrt{2}}|a\rangle(|001\rangle|1\rangle + |011\rangle|1\rangle \\ &\quad + |000\rangle|1\rangle + |010\rangle|1\rangle \\ &\quad + |100\rangle|0\rangle + |101\rangle|0\rangle \\ &\quad + |110\rangle|0\rangle + |111\rangle|0\rangle) \end{aligned} \quad (19e)$$

$$\begin{aligned} |\Psi_3\rangle &= \frac{1}{2\sqrt{2}}|a\rangle(|111\rangle|1\rangle + |011\rangle|1\rangle \\ &\quad + |100\rangle|1\rangle + |000\rangle|1\rangle \\ &\quad + |101\rangle|1\rangle + |001\rangle|1\rangle \\ &\quad + |110\rangle|1\rangle + |010\rangle|1\rangle) \\ &= \frac{1}{2\sqrt{2}}|a\rangle(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)|1\rangle \end{aligned} \quad (19f)$$

A measure on flag qubit gives the desired information.

3 Concise algorithm for quantum associative memories

3.1 Principles of algorithm

We briefly describe here all the process of the quantum associative memory with nonlinear search algorithm. Like in the Rigui paper *et al.* [1] the process of learning or storing patterns of our memory is done using an operator named BDD obtained using the Binary Superposed Quantum Decision Diagram (BSQDD) proposed by Rosenbaum [2] with however any basis states $|z\rangle$ of Hilbert space of 2^n dimensions (not only $|00\dots 0\rangle$).

The process of retrieving data is done by quantum nonlinear search algorithm which allow us to have information we want after a measure on the flag qubit, not on the first register. However it can be useful to measure the register, especially in case of multi-values which satisfy $f(x) = 1$. But, as it appears in the previous section, we will get each 2^n value with the same probability. In the method gives by Rigui *et al.* [1] there is some ambiguities on how the system evolves and it is not clear on how a measure will give one of desired patterns after the retrieving process.

The figure 4 summarizes our quantum associative memory with the nonlinear search algorithm, where it is possible to retrieve one of the desired states in multi-values retrieving, when a measure on the first register is done.

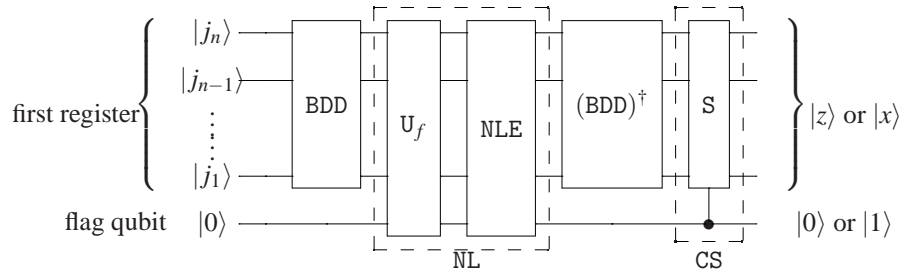


Figure 4: Schematic structure of quantum associative memory with the nonlinear search algorithm. The BDD computes the learning process. The retrieving process is made by the gate U_f which marks the desired states, the gate NLE which computes the repeated nonlinear evolution, the gate $(BDD)^\dagger$ which brings back the first register to its initial state and the conditional operator CS which maps the first register to the needed state $|x\rangle$.

1. The **learning process** is made by the operator BDD.
2. The **retrieving process** is made by:
 - (a) The operator NL which marks the desired states with U_f , computes repeatedly the nonlinear evolution NLE on the system and disentangles the first register from the flag qubit.
 - (b) The operator $(BDD)^\dagger$ which acts on the first register and brings it back to its initial state.

(c) The operator

$$CS = \mathbb{I}_{2^n} \otimes |0\rangle\langle 0| + \left(\sum_{y \neq z} |y\rangle\langle y| + |x\rangle\langle z| + |z\rangle\langle x| - |x\rangle\langle x| \right) \otimes |1\rangle\langle 1|. \quad (20)$$

which is a $(2^{n+1}) \times (2^{n+1})$ conditional operator which maps the first register to the needed state $|x\rangle$ when the flag qubit is $|1\rangle$. In others words,

- if the flag qubit is $|0\rangle$ nothing is done;
- if the flag qubit is $|1\rangle$ the $2^n \times 2^n$ operator

$$\begin{aligned} S &= \mathbb{I}_{2^n} - (|z\rangle\langle z| + |x\rangle\langle x|) + |x\rangle\langle z| + |z\rangle\langle x| \\ &= \sum_{y \neq z} |y\rangle\langle y| + |x\rangle\langle z| + |z\rangle\langle x| - |x\rangle\langle x|. \end{aligned} \quad (21)$$

is applied on the first register.

It is noteworthy that in the case of multi-patterns retrieving the needed state $|x\rangle$ can be a supposed state of all the desired states (for example $|x\rangle = \frac{1}{\sqrt{2}}(|0010\rangle + |1010\rangle)$ in the example 2).

(d) The system is observed by making a measurement on the first register and/or on the flag qubit to erase any ambiguity.

We also point out the fact that as the BSQDD method can compute any desired state, it can be useful to compute the operator CS . Indeed, in the case of a complex needed state $|x\rangle$, where Hadamard gates or others methods are inadequate, the BSQDD method can allow us to have the appropriate form of the operator S .

Example 3. The example 1 suggest that the operator $S = \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes X \otimes \mathbb{I}_2$. Therefore, the CX gate acts only on the second qubit, while the first, third and fourth qubits are unchanged. The figure 5 gives evolution of the system.

3.2 Analysis of the complexity of the nonlinear evolution algorithm

All the above description was made with the assumption that there is at most one value x for which $f(x) = 1$. Let us now consider the case where there can be more than one value satisfying $f(x) = 1$. In the simple case where there are at most two values satisfying $f(x) = 1$, the state (6) must be view like this

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2^n}} \left[\sum_{\substack{j_n j_{n-1} \dots j_1=0 \\ j_n j_{n-1} \dots j_2 \neq i_n i_{n-1} \dots i_2 \\ j_n j_{n-1} \dots j_2 \neq e_n e_{n-1} \dots e_2}}^1 |j_n j_{n-1} \dots j_1\rangle |0\rangle + |i_n i_{n-1} \dots (1 - i_1)\rangle |0\rangle + |i_n i_{n-1} \dots i_1\rangle |1\rangle \right. \\ &\quad \left. + |e_n e_{n-1} \dots (1 - e_1)\rangle |0\rangle + |e_n e_{n-1} \dots e_1\rangle |1\rangle \right]. \end{aligned} \quad (22)$$

Highlighting the LSQ of the first register and the flag qubit, the second part of state (22) must be in one of the following states:

be repeated $(n - r)$ times, thus the number of steps of the quantum associative memory with the nonlinear search algorithm is also

$$\mathcal{O}(n - r). \quad (26)$$

Example 4. For the sake of facility of the comprehension let us consider the parameters of example 1. The marked states are $|2\rangle = |0010\rangle$, $|5\rangle = |0101\rangle$, $|8\rangle = |1000\rangle$, $|10\rangle = |1010\rangle$, $|11\rangle = |1011\rangle$, $|13\rangle = |1101\rangle$ and $|15\rangle = |1111\rangle$. The number of marked states is $m = 7$ and $\log_2 m = 2.807$, consequently $r = 2$ and the NLE gate action starts on the third LSQ of the register. Let us check it in details. As in the example 1 we start with

$$|\psi\rangle = \frac{1}{4}(|0000\rangle + |0001\rangle + |0010\rangle + |0011\rangle + |0100\rangle + |0101\rangle + |0110\rangle + |0111\rangle + |1000\rangle + |1001\rangle + |1010\rangle + |1011\rangle + |1100\rangle + |1101\rangle + |1110\rangle + |1111\rangle)|0\rangle. \quad (27a)$$

The action of the oracle operator \mathbb{U}_f yields to

$$|\Psi\rangle = \frac{1}{4}(|0000\rangle|0\rangle + |0001\rangle|0\rangle + |0010\rangle|1\rangle + |0011\rangle|0\rangle + |0100\rangle|0\rangle + |0101\rangle|1\rangle + |0110\rangle|0\rangle + |0111\rangle|0\rangle + |1000\rangle|1\rangle + |1001\rangle|0\rangle + |1010\rangle|1\rangle + |1011\rangle|1\rangle + |1100\rangle|0\rangle + |1101\rangle|1\rangle + |1110\rangle|0\rangle + |1111\rangle|1\rangle). \quad (27b)$$

Highlighting the third qubit,

and the applying the NLE gate the first time produces:

$$|\Psi\rangle = \frac{1}{4}(|0\boxed{0}00\rangle|\boxed{0}\rangle + |0\boxed{1}00\rangle|\boxed{0}\rangle + |0\boxed{0}01\rangle|\boxed{0}\rangle + |0\boxed{1}01\rangle|\boxed{1}\rangle + |0\boxed{0}10\rangle|\boxed{1}\rangle + |0\boxed{1}10\rangle|\boxed{0}\rangle + |0\boxed{0}11\rangle|\boxed{0}\rangle + |0\boxed{1}11\rangle|\boxed{0}\rangle + |1\boxed{0}01\rangle|\boxed{0}\rangle + |1\boxed{1}01\rangle|\boxed{1}\rangle + |1\boxed{0}11\rangle|\boxed{1}\rangle + |1\boxed{1}11\rangle|\boxed{1}\rangle + |1\boxed{0}00\rangle|\boxed{1}\rangle + |1\boxed{1}00\rangle|\boxed{0}\rangle + |1\boxed{0}10\rangle|\boxed{1}\rangle + |1\boxed{1}10\rangle|\boxed{0}\rangle) \quad (28a)$$

$$|\Psi_1\rangle = \frac{1}{4}(|0\boxed{0}00\rangle|\boxed{0}\rangle + |0\boxed{1}00\rangle|\boxed{0}\rangle + |0\boxed{0}01\rangle|\boxed{1}\rangle + |0\boxed{1}01\rangle|\boxed{1}\rangle + |0\boxed{0}10\rangle|\boxed{1}\rangle + |0\boxed{1}10\rangle|\boxed{1}\rangle + |0\boxed{0}11\rangle|\boxed{0}\rangle + |0\boxed{1}11\rangle|\boxed{0}\rangle + |1\boxed{0}01\rangle|\boxed{1}\rangle + |1\boxed{1}01\rangle|\boxed{1}\rangle + |1\boxed{0}11\rangle|\boxed{1}\rangle + |1\boxed{1}11\rangle|\boxed{1}\rangle + |1\boxed{0}00\rangle|\boxed{1}\rangle + |1\boxed{1}00\rangle|\boxed{1}\rangle + |1\boxed{0}10\rangle|\boxed{1}\rangle + |1\boxed{1}10\rangle|\boxed{1}\rangle) \quad (28b)$$

Finally looking on the most significant qubit,

$$\begin{aligned}
|\Psi_1\rangle = & \frac{1}{4}(|\boxed{0}000\rangle|\boxed{0}\rangle + |\boxed{1}000\rangle|\boxed{1}\rangle \\
& + |\boxed{0}001\rangle|\boxed{1}\rangle + |\boxed{1}001\rangle|\boxed{0}\rangle \\
& + |\boxed{0}010\rangle|\boxed{1}\rangle + |\boxed{1}010\rangle|\boxed{0}\rangle \\
& + |\boxed{0}011\rangle|\boxed{0}\rangle + |\boxed{1}011\rangle|\boxed{1}\rangle \\
& + |\boxed{0}100\rangle|\boxed{0}\rangle + |\boxed{1}100\rangle|\boxed{1}\rangle \\
& + |\boxed{0}101\rangle|\boxed{1}\rangle + |\boxed{1}101\rangle|\boxed{0}\rangle \\
& + |\boxed{0}110\rangle|\boxed{1}\rangle + |\boxed{1}110\rangle|\boxed{0}\rangle \\
& + |\boxed{0}111\rangle|\boxed{0}\rangle + |\boxed{1}111\rangle|\boxed{1}\rangle)
\end{aligned} \tag{28c}$$

and applying the NLE gate the last time produces:

$$\begin{aligned}
|\Psi_2\rangle = & \frac{1}{4}(|\boxed{0}000\rangle + |\boxed{1}000\rangle \\
& + |\boxed{0}001\rangle + |\boxed{1}001\rangle \\
& + |\boxed{0}010\rangle + |\boxed{1}010\rangle \\
& + |\boxed{0}011\rangle + |\boxed{1}011\rangle \\
& + |\boxed{0}100\rangle + |\boxed{1}100\rangle \\
& + |\boxed{0}101\rangle + |\boxed{1}101\rangle \\
& + |\boxed{0}110\rangle + |\boxed{1}110\rangle \\
& + |\boxed{0}111\rangle + |\boxed{1}111\rangle)|\boxed{1}\rangle
\end{aligned} \tag{28d}$$

It effectively appears that the NLE gate was repeated $(n - r) = 4 - 2 = 2$ times.

If we know the values of t qubits of our first register (i.e., t qubits have been measure or are already disentangled to others, or the oracle acts on a subspace of $(n - t)$ qubits) and there is at most m values for which $f(x) = 1$, the NLE gate will acts repeatedly $((n - t) - r)$ times, where $r = \text{int}(\log_2 m)$, starting on the $(r + 1)^{\text{th}}$ LSQ. As the t qubits which are already known will be ignored, it is clear that $m \leq 2^{n-t}$. Consequently the number of steps of the quantum associative memory with the nonlinear search algorithm is

$$\mathcal{O}((n - t) - r). \tag{29}$$

Now if in the first register, which is a n -qubit system, the computed patterns are $p \leq 2^n$ and we stated $b = \text{ceil}(\log_2 p)$, i.e., the least integer greater or equal to $\log_2 p$, m the number of value x for which $f(x) = 1$, and $r = \text{int}(\log_2 m)$. The the NLE gate will acts repeatedly $(b - r)$ times. Therefore, the number of steps of the quantum associative memory with the nonlinear search algorithm is

$$\mathcal{O}(b - r), \tag{30}$$

for which the upper bound is Eq.(26). Note that the starting point of the NLE gate action will always be the $(r + 1)^{\text{th}}$ LSQ.

If we know the values of t qubits of our first register, it suppose that we must view the system in terms of $q \leq p \leq 2^n$ patterns; consequently the number of value m for which $f(x) = 1$ is $m \leq q$. For $c = \text{ceil}(\log_2 q)$, the NLE gate will acts repeatedly $(c - r)$ times. Therefore, the general form of number of steps of the quantum associative memory with the nonlinear search algorithm is

$$\mathcal{O}(c - r). \tag{31}$$

Example 5. • If we consider again the parameters of the example 1, we find that $p = 16$, the number of known qubits is $t = 0$. Consequently $q = p = 16$, then $\log_2 q = \log_2 16 = 4.0$, thus $c = 4$. $m = 1$, then $r = 0$. The NLE gate will acts repeatedly $(c - r) = 4 - 0 = 4$ times.

- In the example 4, $t = 0$, $q = 16$ but $m = 7$. Then $\log_2 m = 2.807$, that is $r = 2$. The NLE gate will acts repeatedly $(c - r) = 4 - 2 = 2$ times.
- In the example 2, $t = 1$, consequently $q = 8$ according to the assumption take in this example. $\log_2 q = \log_2 8 = 3.0$, thus $c = 3$. $m = 1$, then $r = 0$. The NLE gate will acts repeatedly $(c - r) = 3 - 0 = 3$ times.

It is noteworthy that when the state $|01\rangle + |11\rangle$ appears the last time the NLE gate acts repeatedly, it not evolves. Its nonlinear evolution must be like that of the state (8a) and acts like follows:

Step 4. Apply operator U:

$$U(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle - |11\rangle). \quad (32)$$

Step 5.1. Apply the nonlinear operator NL^- :

$$NL^- \left[\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \right] = \sqrt{2}|0\rangle(\delta|0\rangle + \varepsilon|1\rangle), \quad (33)$$

where $\delta, \varepsilon \in \mathbb{C}$, $|\delta|^2 + |\varepsilon|^2 = 1$. As we see on the state (33), the action of the nonlinear operator NL^- is also not specify like on the state (11a).

Step 5.2. Apply the 1-qubit nonlinear operator NL^+ on the flag qubit:

$$NL^+[\sqrt{2}|0\rangle(\delta|0\rangle + \varepsilon|1\rangle)] = \sqrt{2}|00\rangle. \quad (34)$$

The general form of the unitary matrix NL^+ which transforms the generic one-qubit $\delta|0\rangle + \varepsilon|1\rangle$ to $|0\rangle$ is

$$\Pi = \begin{pmatrix} \delta^* & \varepsilon^* \\ \mp\gamma\varepsilon & \pm\gamma\delta \end{pmatrix}, \quad \gamma = \pm i \text{ or } \pm 1 \ (i^2 = -1), \quad (35)$$

where $\delta, \varepsilon \in \mathbb{C}$, $|\delta|^2 + |\varepsilon|^2 = 1$.

Step 6. Apply the NOT gate X on the flag qubit and the Hadamard gate W on the first qubit.

We summarize below this nonlinear evolution of state $|01\rangle + |11\rangle$ with its corresponding circuit:

$$|01\rangle + |11\rangle \xrightarrow{U} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \xrightarrow{NL^-} \sqrt{2}|0\rangle(\delta|0\rangle + \varepsilon|1\rangle) \xrightarrow{NL^+} \sqrt{2}|00\rangle \xrightarrow{W \otimes X} |01\rangle + |11\rangle \quad (36)$$

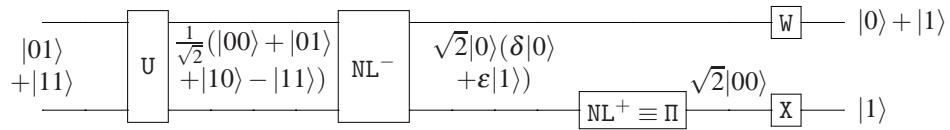


Figure 6: Equivalent circuit of the nonlinear evolution (36).

Finally, the quantum associative memory with the nonlinear search can be describe by the algorithm (2), where

- n is the number of qubit of the first register;
- $p \leq 2^n$ the number of stored patterns;
- $q \leq p$ the number of stored patterns if the value of t are known (i.e., t qubits have been measure or are already disentangled to others, or the oracle acts on a subspace of $(n - t)$ qubits);
- $c = \text{ceil}(\log_2 q)$, i.e., the least integer greater or equal to $\log_2 q$;
- $m \leq q$ the number of values x for which $f(x) = 1$;
- $r = \text{int}(\log_2 m)$ is the integer part of $\log_2 m$.

Algorithm 2 Algorithm of quantum associative memory with nonlinear search algorithm

- 1: Store patterns using the operator BDD and put the flag qubit to $|0\rangle$
 - 2: Apply the oracle U_f
 - 3: Repeat $(c - r)$ times step (4) to step (6) (i.e., one time per qubit of the first register, starting from $(r + 1)^{th}$ qubit, with the flag qubit)
 - 4: apply the unitary operator U
 - 5: 1. apply the nonlinear operator NL^-
 2. apply the nonlinear operator NL^+
 - 6: apply the Hadamard operator W on the qubit of the first register and the NOT operator X on the flag qubit
 - 7: Apply the operator $(BDD)^\dagger$ to kick back the first register to its initial state
 - 8: Apply the conditional operator CS to flip the register to needed state
 - 9: Observe the system
-

4 Conclusion

We have proposed a model of the Quantum Associative Neural Network with the Nonlinear Search Algorithm similar to that of Rigui *et al.* [1], with however the possibility to retrieve one of desired states in multi-values retrieving, when a measure on the first register is done.

Firstly we have described the Nonlinear Search Algorithm put forth by Abrams and Llyod in [3] with notations that overcome some ambiguities due to the notations of Rigui *et al.* and Czachor[7] and by summarizing each step of the nonlinear evolution with an equivalent circuit. A good general form of the unitary matrix NL^+ which acts on the generic flag qubit $\alpha|0\rangle + \beta|1\rangle$ was given, thereby correcting the wrong one given by Rigui *et al.* .

Secondly we have described our model of the Quantum Associative Neural Network where we have introduce a $(2^{n+1}) \times (2^{n+1})$ conditional operator, CS, which maps the first register to the needed state $|x\rangle$ when the flag qubit is $|1\rangle$, where n is the number of qubit of the first register. If n is the number of qubit of first register, $p \leq 2^n$ the number of stored patterns, $q \leq p$ the number of stored patterns if the value of t are known (i.e., t qubits have been measure or are already be disentangled to others, or the oracle

acts on a subspace of $(n-t)$ qubits), $m \leq q$ the number of values x for which $f(x) = 1$, $c = \text{ceil}(\log_2 q)$ the least integer greater or equal to $\log_2 q$, and $r = \text{int}(\log_2 m)$ the integer part of $\log_2 m$, then the time complexity of our algorithm is $\mathcal{O}(c-r)$. It is better than Grover's algorithm and its modified forms which need $\mathcal{O}(\sqrt{\frac{2^n}{m}})$ steps, when they are used as the retrieval algorithm. An example to illustrate the results give by our analysis was done. It noteworthy that our algorithm also allows to measure the flag qubit to erase any ambiguities on the result gives by a measurement on the first register.

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